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A NOTE ON MINIMAL MATRIX REPRESENTATION OF CLOSURE OPERATIONS

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A matrix M with n columns represents a closure operation $F(A)$, $(A\subset X,\ |X|=n)$ if for any A, any two rows equal in the columns corresponding to A are also equal in $F(A)$. Let $m(F)$ be the minimum number of rows of the matrices representing F. Lower and upper estimates on max $m(F)$ are given where max runs over the set of all closure operations on n elements.

Put $X=\{1, 2, ..., n\}$. A function $F: 2^x \to 2^x$ satisfying the conditions

(1) $$A \subseteqq F(A),$$

(2) $$A \subseteqq B \Rightarrow F(A) \subseteqq F(B),$$

(3) $$F(F(A)) = F(A), \quad A, B \subseteqq X$$

is called a *closure operation*. If M is an $m\times n$ matrix then $F_M(A)$, $A\subseteqq X$, contains the ith column of M iff

(4) any two rows identical in columns belonging to A are also equal in the ith column.

It is easy to see that $F_M(A)$ is a closure operation. We say that M *represents* the closure operation F if $F=F_M$. It is known [1] that any closure operation is representable with an appropriate matrix M. Let $m(F)$ denote the minimum number of rows of these M. Finally, let $R(n)=\max m(F)$ where the maximum is taken over all the closure operations on n elements.

If F is a closure operation define

$$\mathscr{K}_F = \{A\colon F(A) = X \text{ and } A \text{ is minimal for this property}\}.$$

We say that an $m\times n$ matrix M represents the family $\mathscr{K}$ iff $\mathscr{K}=\mathscr{K}_{F_M}$. Let $m(\mathscr{K})$ denote the minimum number of rows of these M. Finally, let $r(n)=\max m(\mathscr{K})$ where the maximum is taken over all such possible families $\mathscr{K}$ on n elements.

Observe that $m(\mathscr{K}_F)\leqq m(F)$ holds for any F. Hence, by definition,

(5) $$r(n) \leqq R(n)$$

AMS subject classification (1980): 01 A 10 follows. On the other hand [2] proves the theorem

(6) $$\frac{1}{n^2}\binom{n}{\lfloor n/2\rfloor} \leqq r(n) \leqq \binom{n}{\lfloor n/2\rfloor}+1.$$

The aim of this paper is to prove the analogous inequalities for $R(n)$:

Theorem.

(7) $$\frac{1}{n^2}\binom{n}{\lfloor n/2\rfloor} \leqq R(n) \leqq \left(1+C\left(\frac{1}{\sqrt{n}}\right)\right)\binom{n}{\lfloor n/2\rfloor}.$$

Proof. The left hand side follows from (6) and (5). To prove the right hand side we construct a matrix M with this many rows for any given closure operation F.

Let $\mathscr{B}_F=\{A\colon F(A)=A\}$. It is easy to see [1] that $\mathscr{B}_F$ has the property that $A, B\in\mathscr{B}_F \Rightarrow A\cap B\in\mathscr{B}_F$ and conversely if $\mathscr{B}$ posesses this property then there is an F with $\mathscr{B}_F=\mathscr{B}$ (F is defined by $F(A)=\cap B$ where the intersection is taken over all sets B satisfying $A\subseteqq B\in\mathscr{B}$). If $\mathscr{A}$ is any family of sets let $\tilde{\mathscr{A}}$ denote the family of all possible intersections of its members. Given F, we form a sequence $\mathscr{B}_F=\mathscr{B}_0\supseteqq\supseteqq\mathscr{B}_1\supseteqq\mathscr{B}_2\supseteqq\ldots\supseteqq\mathscr{B}_k$ of families. Choose three different members (if there exist) A, B and C of $\mathscr{B}_i$ satisfying $A\cap B=C$. $\mathscr{B}_{i+1}$ is defined by $\mathscr{B}_{i+1}=\mathscr{B}_i-\{C\}$. The statement $\tilde{\mathscr{B}}_{i+1}=\tilde{\mathscr{B}}_i$ is trivial. Therefore, $\tilde{\mathscr{B}}_i=\tilde{\mathscr{B}}_0=\mathscr{B}_F$ holds for any i $(0\leqq i\leqq k)$. It is easy to see that the procedure stops after a finite number of steps: in $\mathscr{B}_k$ there are no 3 different members with $A\cap B=C$. A theorem of Kleitman [3] states that the size of a family of subsets of an n element set containing no 3 different members with $A\cap B=C$ cannot exceed $\left(1+O\left(\frac{1}{\sqrt{n}}\right)\right)\binom{n}{\lfloor n/2\rfloor}$.

We construct M using the family $\mathscr{B}_k=\{A_1,\ldots,A_s\}$. The jth $(1\leqq j\leqq n)$ entry of the ith $(0\leqq i\leqq s)$ row is

$$a_{ij}=\begin{cases}0 & \text{if} \quad i=0\\ 0 & \text{if} \quad 1\leqq i\leqq s, \quad j\in A_i\\ i & \text{if} \quad 1\leqq i\leqq s, \quad j\notin A_i.\end{cases}$$

The number of rows of M is $1+|\mathscr{B}_k|\leqq\left(1+O\left(\frac{1}{\sqrt{n}}\right)\right)\binom{n}{\lfloor n/2\rfloor}$. We have to prove only that M represents F. Let A be an arbitrary subset of X. Suppose first that $j\in F(A)$. Two rows i_1, i_2 can be equal in the columns corresponding to A only then if all these entries are zeros. That is, if $A\subseteqq A_{i_1}, A_{i_2}$, (2) implies $F(A)\subseteqq F(A_{i_1})$, $F(A_{i_2})$. As $A_{i_1}, A_{i_2}\in\mathscr{B}_F$, $F(A)\subseteqq A_{i_1}$, A_{i_2} follows. Consequently, the two rows are equal $(0=0)$ in the jth column.

Suppose now $j\notin F(A)$, $F(A)\in\mathscr{B}_F$ holds by (3). $\tilde{\mathscr{B}}_k=\mathscr{B}_F$ implies that $F(A)$ is an intersection of some $(\geqq 1)$ members of $\mathscr{B}_k$. One of these members, say A_i does not contain j. The 0th and the ith rows are equal in A but are different in the jth column. That is, $F_M=F$ holds. ∎

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References

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